Ec140 - Conditional Probability and Expectation

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Housekeeping

- Unofficial Course Capture is now live.
- This is the last dry-math (without context) lecture!
- After I hope that we will be able to do a in-depth read of MM.

Today's Lecture

- Conditional Probability
- Conditional Expectation

Conditional Probability: Definition

- The **probability distribution** of a random variable Y given that we observe a the **value** of another random variable X, is the probability that we observe both events, re-scaled by the probability of the event we observe.

$$P(Y = y|X = x) = \frac{P(Y = y \text{ and } X = x)}{P(X = x)}$$

Conditional Probability: Definition

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$$P(Y = y|X = x) = \frac{P(Y = y, X = x)}{P(X = x)}$$

- In textbooks, you will probably see this definition in terms of events. Let A and B denote two random events. Then $P(A|B) = \frac{P(A,B)}{P(B)}$

Conditional Probability: Intuition With Data 1/4

Given data on passing status and whether students submitted all their assignments (PS, Readings, Midterms, Exam), you want to know what is the probability of passing **conditional** on submitting everything? (Why does this matter?)

$$P(Y = y|X = x) = \frac{P(Y = y, X = x)}{P(X = x)}$$

Let's re-write it using the current random variables:

i	Passed Course ($Pass$)	Submited Everything (S)
1	1	1
2	1	0
3	1	1
4	0	0
5	1	1
6	0	1
7	1	1
8	1	1
9	1	1

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$$P(Y = y|X = x) = \frac{P(Y = y, X = x)}{P(X = x)}$$

Let's re-write it using the current random variables:

$$P(Pass = 1|S = 1) = \frac{P(Pass = 1, S = 1)}{P(S = 1)}$$

i	**Passed Course ($Pass$)**	**Submited Everything (S)**
1	1	1
2	1	0
3	1	1
4	0	0
5	1	1
6	0	1
7	1	1
8	1	1
9	1	1

Conditional Probability: Intuition With Data 2/4

$$P(Pass = 1|S = 1) = \frac{P(Pass = 1, S = 1)}{P(S = 1)}$$

- How would you construct the data for $P(Pass = 1|S = 1)$?
    - What do you think about rows `2` and `4`?

i	$Pass$	S
1	1	1
2	1	0
3	1	1
4	0	0
5	1	1
6	0	1
7	1	1
8	1	1
9	1	1
10	1	1

Conditional Probability: Intuition With Data 2/4

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- How would you construct the data for $P(Pass = 1|S = 1)$?
  - What do you think about rows `2` and `4`?

i	$Pass$	S	$Pass\|S$
1	1	1	1
2	1	0	NA
3	1	1	1
4	0	0	NA
5	1	1	1
6	0	1	0
7	1	1	1
8	1	1	1
9	1	1	1
10	1	1	1

Conditional Probability: Intuition With Data 2/4

$$P(Pass = 1|S = 1) = \frac{P(Pass = 1, S = 1)}{P(S = 1)}$$

- How would you construct the data for $P(Pass = 1|S = 1)$?
  - What do you think about rows `2` and `4`?

$$P(Pass = 1|S = 1) = \frac{\sum_i \#(pass_i = 1|s_i = 1)}{8} = 0.875$$

i	$Pass$	S	$Pass\|S$
1	1	1	1
2	1	0	NA
3	1	1	1
4	0	0	NA
5	1	1	1
6	0	1	0
7	1	1	1
8	1	1	1
9	1	1	1
10	1	1	1

Conditional Probability: Intuition With Data 3/4

$$P(Pass = 1|S = 1) = \frac{P(Pass = 1, S = 1)}{P(S = 1)}$$

- $P(Pass = 1, S = 1) = 0.7$
- $P(S = 1) = 0.8$

$$\frac{P(Pass = 1, S = 1)}{P(S = 1)} = 0.875$$

- Same as $P(Pass = 1|S = 1)$
-
    - Draw histogram for $(Pass, S)$

i	$Pass$	S	$Pass\|S$	$Pass, S$
1	1	1	1	1
2	1	0	NA	0
3	1	1	1	1
4	0	0	NA	0
5	1	1	1	1
6	0	1	0	0
7	1	1	1	1
8	1	1	1	1
9	1	1	1	1
10	1	1	1	1

Conditional Probability: Intuition With Data 4/4

- A key step in conditioning is to remember to re-scale the probabilities
- $P(Pass = 1) = 0.8$, while $P(Pass = 1|S = 1) = 0.875$ so knowing about S changed the distribution of $P(Pass = 1)$, this is the opposite of what concept we discussed last class?
- (You can see this additional great intuition based on events)

Conditional Probability: Bayes Rule 1/3

$$P(Y = y|X = x) = \frac{P(Y = y, X = x)}{P(X = x)}$$

- There is a lot of multiplication in this step, so let me replace the notation $P(Y = y)$ with just $P(Y)$.

$$P(Y|X) = \frac{P(Y, X)}{P(X)}$$

Conditional Probability: Bayes Rule 2/3

$$P(Y|X) = \frac{P(Y,X)}{P(X)}$$

Notice

$$P(X|Y) = \frac{P(Y,X)}{P(Y)} \Rightarrow P(X|Y)P(Y) = P(Y,X)$$

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

Conditional Probability: Bayes Rule 3/3

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

- One famous problem, to practice the concepts above, it the Monty Hall problem. See this explanation by Berkeley's Lisa Goldberg: intuition, math
- (This equation provides a prescription for rational, open minded thinking. It basically guides us on how to update our beliefs about something (Y), after observing some evidence (X) .)
- (Our updated beliefs $(P(Y|X))$ should be equal to our previous beliefs $(P(Y))$ times the probability that the evidence we observe (X) is consistent with the thing we are interested in $(P(X|Y))$, scaled by the probability of observing the evidence $(P(X))$.)

Conditional Probability: Break Probabilities into Pieces 1/2

- Draw blob of event B in some event space. Cut it into pieces that don't intersect with each outer (disjoint sets)

- We can compute the probability of B as follows:

$$P(B) = P(B, A_1) + P(B, A_2)$$

Conditional Probability: Break Probabilities into Pieces 2/2

$$P(B) = P(B, A_1) + P(B, A_2)$$

- But we know have a handy expression for the probability of two events $(P(Y, X) = P(Y|X)P(X))$. Hence:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$$

- We can do the same with many pieces $(A_1, A_2, \ldots, A_J)$, so a general expression would be:

$$P(B) = \sum_i P(B|A_i)P(A_i)$$

- This expression is known as the **law of total probabilities**

Activity 1: 1/3

- Given that this video is more complex that the previous activities, I'll stop in different parts as you about it (It looks pretty innocent but I will need your full attention)

- Watch this video from Stat 110. And answer the following questions:

[after "95%"]

- What are the random variables in this problem? What are they mapping into instead of numbers?

[after "5% misdiagnosis in each case"]

Activity 1: 2/3

[after "5% misdiagnosis in each case"]

- Let's use S for sick r.v. (1 = sick, 0 = healthy) and T for test positive r.v. (1 = test positive, 0 = test negative). Write down the two 95% described here as conditional probabilities.
  - Upper branch: $P(T=1|S=1)=95\%$ and $P(T=0|S=1)=5\%$, lower branch: $P(T=0|S=0)=95\%$ and $P(T=1|S=0)=5\%$

["How sure are you"]

Activity 1: 3/3

["Correctly tested as negatives]

- Where does the 1881 comes from?

["Falsely tested as negative"]

- Where does the 1 comes from?

['After the 16%"]

- Use bayes rule and the law of total probabilities to obtain the 16%

Activity 1: 3/3

- Use bayes rule and the law of total probabilities to obtain the 16%
    - We want to know the probability that Jimmy is sick, given that he tested positive:

$$P(S=1|T=1) = \frac{P(T=1|S=1)P(S=1)}{P(T=1)}$$

We know that the overall probability getting sick is 1%, and that $P(T=1|S=1)=95\%$. But we do not know the overall probability of testing positive, for this we can use the law of total probability:

$$P(T=1) = P(T=1|S=1)P(S=1) + P(T=1|S=0)P(S=0)$$
$$= 0.95 \times 0.01 + 0.05 \times 0.99 = 0.059$$

Hence:

$$P(S=1|T=1) = \frac{0.95 \times 0.01}{0.059} = 0.161$$

Conditional Expectation: Definition

Remember the definition of expected value:

$$\mathbb{E}(X) = \sum_{x} xP(X = x)$$

We can do the same for another random variable Y:

$$\mathbb{E}(Y) = \sum_{y} yP(Y = y)$$

If we want to know the $\mathbb{E}(Y|X)$ we need to use the definition of expectation for Y, but using the appropriate probabilities:

$$\mathbb{E}(Y|X) = \sum_{y} yP(Y = y|X = x)$$

Conditional Expectation: Definition

- This is the most important expression for the rest of the course.
- With it, we will study randomized controlled trials, regression, and everything else!

$$\mathbb{E}(Y|X) = \sum_y yP(Y = y|X = x)$$

Conditional Expectation: Activity 2 1/2

Our Last Stat 110 Video!

- Discuss a plausible explanation for the law of iterated expectations (Adam's Law):

$$\mathbb{E}(Y) = \mathbb{E}_{\mathrm{x}}(\mathbb{E}_{\mathrm{y}}(Y|X)) = \sum_x E(Y|X=x)P(X=x)$$

- (for those interested in the full proof, see here)
- What is changing when moving the "city's pile"?

Conditional Expectation: Activity 2 2/2

- For the the variance of Y in terms of conditionals of X (Eve's Law). Which term corresponds to the between variation? Which to the within? Between and within what?

$$\mathbb{V}(Y) = \mathbb{E}(\mathbb{V}(Y|X)) + \mathbb{V}(\mathbb{E}(Y|X))$$

- Where does the 1400 comes from?

Acknowledgments

- Stat 110
- Nick HK
- Numberphile (Lisa Goldberg)